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Two-Step Method for Evolving Nonlinear Acoustic Systems to a Steady State

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A two-step method for evolving two-dimensional nonlinear acoustic systems with flow to a periodic steady state is presented. In the first step of the method, the full nonlinear system governing acoustic disturbances is integrated numerically starting with arbitrary initial conditions using the explicit predictor-corrector method developed by MacCormack. In the second step, the Fourier Time Transform of the computed field is calculated to determine its frequency components. The transient wave field is then filtered from the spectrum and the inverse transform taken to provide an approximation to the steady-state wave field. This approximate field provides a new initial condition for subsequent iterations on the method. The method is tested on a benchmark acoustic problem for which exact steady-state solutions are known and on a nonlinear problem for which a steady-state solution has not been given before. Excellent agreement with the benchmark acoustic solution was obtained within four iterations for planar and nonplanar sources. Convergence to a steady state for the nonlinear problem occurred in six iterations. The two-step method eliminates the need to develop nonreflecting boundary conditions in order to obtain periodic steady-state solutions of two-dimensional acoustic problems and is easily extended to any number of spatial dimensions and to other hyperbolic systems. The procedure is shown to be numerically stable and may provide the only alternative for obtaining steady-state solutions to problems for which nonreflecting boundary conditions are not known or are physically incorrect.

Introduction

A SOLUTION to the equations of acoustics with flow is necessary to understand the interaction between sound and fluid motion. No exact solutions to the acoustic equations in their most general nonlinear form are available, and none is likely to be found in the foreseeable future. Therefore, since the early 1970s growing attention has been paid to the numerical solution of these equations for internal flows. Most of the physical problems modeled have involved periodic acoustic sources since they are common in practice. In real fluid media these give rise to periodic steady-state acoustic fields in which transient waves carrying information about the initial state have been dissipated by viscous forces. Knowledge of the steady-state field is often the primary goal of acoustic analyses.

Computational techniques for achieving this goal have proceeded along two lines. One is the classical steady-state theory in which acoustic waves are assumed a priori to be periodic functions of time. To date, this theory has been restricted mostly to linear acoustic disturbances propagating in nonphysical mean flows.[1-5] This approach has several additional shortcomings,[6] and is not the best candidate for installation on the current generation of supercomputers. As an alternative, transient numerical techniques were later developed.[6] In these, the acoustic equations are integrated forward in real time until the solution evolves to a steady state. Is is generally accepted[6] that the transient methods are considerably less expensive than the steady-state methods.

There is, however, a vexing and still open issue in transient numerical analyses regarding how the solution can be evolved to a steady state. In formulating equations of motion for acoustic disturbances, it is often desirable to ignore viscous forces. This simplifies the governing equations and, most importantly, the formulation of the necessary boundary conditions for their solution. In addition, it is usually true that viscosity exerts only minimal influence on the steady-state sound field itself. Unfortunately, with dissipative effects eliminated the transient wave field associated with initial conditions does not decay and may be trapped within the computational domain so that a steady state will not be achieved. Ultimately, the ability of the transient methods to evolve to a steady state will determine much of their usefulness.

Several strategies currently exist for evolving nondissipative hyperbolic systems to a steady state. One strategy is to introduce explicit artificial viscosity into the numerical scheme. This approach works well with steady flows but often leads to excessive damping of acoustic waves. Higher-order time accurate schemes that possess minimal artificial dissipation are better candidates for calculating acoustic fields.[7] One commonly used strategy for obtaining steady-state solutions with such schemes has been to use nonreflecting boundary conditions.[8-11] These have the advantage that they allow the transient field to pass out through the computational boundary without reflecting. On the other hand, if the computational boundaries are not nonreflecting, the transient field will be reflected into the computational domain and a method for extracting the steady-state field from the solution is required.

Nonreflecting boundary conditions are the exception rather than the rule for acoustic problems within finite domains. Many of the most common acoustic boundaries found in nature are actually reflecting. Among these are rigid boundaries and those involving pressure releases. In fact, for

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acoustic problems in more than one space dimension and time, nonreflecting boundary conditions in flow are at best difficult and in most cases impossible to develop.[9] For example, the current state-of-the-art for acoustic propagation in channels is to use the steady-state acoustic impedance for outgoing waves as the nonreflecting boundary condition[6,7] even during the transient state. Unfortunately, the steady-state acoustic impedance cannot be determined in most cases. Furthermore, even when it is known it may lead to reflections of transient waves and to unstable results.

The current paper was motivated by the lack of a method for extracting the steady-state acoustic wave field from transient analyses when nonreflecting boundary conditions are not known, or are physically incorrect. Although the method is presented in two space dimensions and for rectangular geometries, the concept has clear relevance and is easily extended to three-dimensional acoustic propagation and arbitrary geometries. The aim of the current paper is to establish the validity of the method by applying it to a benchmark problem whose solution is well understood analytically. This problem is believed to exemplify aspects of the more complicated acoustic problems which are the real target of this analysis. The method is also applied to a nonlinear problem for which neither an analytical nor a numerical solution has been presented previously.

Equations Governing the Acoustic System

Figure 1 illustrates the computational domain and coordinate system to be used in this analysis. The fluid is considered to be laminar, viscous, compressible, and flowing subsonically from left to right. The inflow and outflow boundaries are located at $x = 0$ and $x = L$, respectively, as shown in the figure, and the upper and lower boundaries are considered to be rigid and nonconducting. It is assumed that for all times $t < 0$ the flow in the channel consists only of a known steady flow. However for times greater than zero, an acoustic pressure source $p_s(y,t)$ is prescribed in addition to the steady flow at the outflow boundary. As is common in acoustics, the source pressure is assumed known for all $t > 0$.

To begin, the fundamental equations governing the fluid flow are two-dimensional Navier-Stokes equations, which are written in conservative form as

$$\frac{\partial\{U\}}{\partial t} + \frac{\partial\{H\}}{\partial x} + \frac{\partial\{G\}}{\partial y} = \{0\} \tag{1}$$

The vector components in Eq. (1) are given in virtually any fluid dynamics text (for example, see Ref. 12) and are not written explicitly. The unknown variables in $\{U\}$ are the fluid density ρ, axial component of velocity u, transverse component of velocity v, fluid temperature T, and pressure p. To derive the equations governing the acoustic system in flow, the unknown fluid variables are decomposed into a mean or steady flow and an acoustic perturbation component

$$\{\Phi\} = \{\Phi_0\} + \{\tilde{\Phi}\}, \qquad \{\Phi\} = \begin{Bmatrix} \rho \\ u \\ v \\ p \end{Bmatrix} \tag{2a}$$

$$\{\Phi_0\} = \begin{Bmatrix} \rho_0 \\ u_0 \\ v_0 \\ p_0 \end{Bmatrix}, \qquad \{\tilde{\Phi}\} = \begin{Bmatrix} \tilde{\rho} \\ \tilde{u} \\ \tilde{v} \\ \tilde{p} \end{Bmatrix} \tag{2b}$$

in which $\{\Phi\}$, $\{\Phi_0\}$, and $\{\tilde{\Phi}\}$ are vectors containing the total, steady, and acoustic variables in the flow, respectively. Furthermore, viscous and heat conduction effects are considered important only on the mean process and an ideal gas is assumed to eliminate temperature from the system. Under these assumptions, Eq. (1) can be written as the following hyperbolic system governing the acoustic perturbation variables:

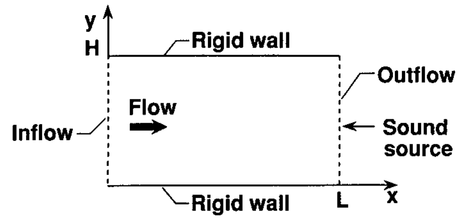

Fig. 1 Rectangular channel and coordinate system.

$$\frac{\partial\{\tilde{\Phi}\}}{\partial t} + [A]\frac{\partial\{\tilde{\Phi}\}}{\partial x} + [B]\frac{\partial\{\tilde{\Phi}\}}{\partial y} + \{\tilde{F}\} = \{0\} \tag{3a}$$

$$[A] = \begin{Bmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & \frac{1}{\rho} \\ 0 & 0 & u & 0 \\ 0 & \rho c^2 & 0 & u \end{Bmatrix} \tag{3b}$$

$$[B] = \begin{Bmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & \frac{1}{\rho} \\ 0 & 0 & \rho c^2 & v \end{Bmatrix} \tag{3c}$$

$$\{\tilde{F}\} = [A]\frac{\partial\{\Phi_0\}}{\partial x} + [B]\frac{\partial\{\Phi_0\}}{\partial y} - \{F_v\} \tag{3d}$$

$$\{F_v\} = \begin{Bmatrix} 0 \\ \left[\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{xy}\right] \\ \left[\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy}\right] \\ (\gamma - 1)\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \tau\right] \end{Bmatrix} \tag{3e}$$

$$\tau = \frac{\partial u_0}{\partial x}\tau_{xx} + \left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right)\tau_{xy} + \frac{\partial v_0}{\partial y}\tau_{yy} \tag{3f}$$

in which $c = (\gamma p/\rho)^{1/2}$ is the local speed of sound. The components of the viscous stress tensor τ_{xx}, τ_{yy}, and τ_{xy}, and the heat flux vector, q_x and q_y, are computed in the usual manner[12] using the steady flow variables.

There exist no exact analytical solutions to the equations of acoustics for an arbitrary steady flow, and they have not yet yielded even to computer solutions in this most general form. Before the system can be solved, a set of boundary conditions consistent with Eq. (3a) must be specified.

Acoustic Boundary Conditions

The boundary conditions used here are taken directly from those of Ref. 7 and are discussed only briefly. To begin, Eq. (3a) is written in the following form:

$$\frac{\partial\{\tilde{\Phi}\}}{\partial t} = \{F_S^+\} + \{F_S^-\} - \{F\} \tag{4a}$$

$$\{F\} = [B]\left\{\frac{\partial\tilde{\Phi}}{\partial y}\right\} + \{\tilde{F}\} \tag{4b}$$

$$\{F_S^+\} = \begin{Bmatrix} \dfrac{(u+c)b_{3S}}{2c^2} - \dfrac{ub_{1S}}{c^2} \\ \dfrac{(u+c)b_{3S}}{2\rho c} \\ \lambda_2 b_{2S} \\ \dfrac{(u+c)b_{3S}}{2} \end{Bmatrix} \tag{4c}$$

$$\{F_S^-\} = \frac{1}{2}\begin{Bmatrix} \dfrac{(u-c)b_{4S}}{c^2} \\ -\dfrac{(u-c)b_{4S}}{\rho c} \\ 0 \\ (u+c)b_{4S} \end{Bmatrix} \tag{4d}$$

$$b_{1S} = -\left[\frac{\partial \tilde{p}}{\partial x} - c^2 \frac{\partial \tilde{\rho}}{\partial x}\right], \qquad b_{2S} = -\frac{\partial \tilde{v}}{\partial x} \tag{4e}$$

$$b_{3S} = -\left[\frac{\partial \tilde{p}}{\partial x} + \rho c \frac{\partial \tilde{u}}{\partial x}\right], \qquad b_{4S} = -\left[\frac{\partial \tilde{p}}{\partial x} - \rho c \frac{\partial \tilde{u}}{\partial x}\right] \tag{4f}$$

In physical terms, b_{1S} and b_{2S} can be identified with entropy and vorticity waves, respectively, while b_{3S} and b_{4S} are associated with acoustic waves propagating in opposite directions. Here, $\{F_S^+\}$ and $\{F_S^-\}$ represent wave components propagating in the positive and negative x direction, respectively, at the inflow-outflow boundaries. As discussed in Ref. 7, three physical quantities must be specified at the inflow boundary for a subsonic flow. Here we specify b_{1S}, the acoustic vorticity ζ, and an acoustic impedance z so that

$$b_{1S} = -\left[\frac{\partial \tilde{p}}{\partial x} - c^2 \frac{\partial \tilde{\rho}}{\partial x}\right] = b(y) \tag{5a}$$

$$b_{2S} = -\left[\zeta + \frac{\partial \tilde{u}}{\partial y}\right], \qquad \zeta = \frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y} \tag{5b}$$

$$b_{3S} = \frac{1}{(u-c)(1-\frac{z}{\rho c})}\left[z_0 - \left(1 + \frac{z}{\rho c}\right)(u+c)b_{4S}\right] \tag{5c}$$

$$z_0 = 2\left(F_4 - zF_2 + \tilde{u}\frac{\partial z}{\partial t}\right), \qquad z = \frac{\tilde{p}(0,y,t)}{\tilde{u}(0,y,t)} \tag{5d}$$

in which F_i denotes the ith component of $\{F\}$. As discussed previously,[7] z will be referred to as the acoustic impedance, although it is not the complex quantity usually defined for steady-state harmonic motion. Furthermore, Eq. (5) in general do not eliminate waves reflected from the inflow boundary. The outflow boundary requires one physical boundary condition for a subsonic flow. Thus at the outflow boundary b_{4S} is specified in terms of the outgoing wave components and the known source pressure p_s as

$$b_{4S} = \frac{1}{(u-c)}\left\{2\left[\frac{\partial p_s}{\partial t}(y,t) + F_4\right] - (u+c)b_{3S}\right\} \tag{6}$$

In the following p_s will be assumed to be a periodic function of time with a known period T.

The upper and lower boundaries, $y=0$ and $y=H$, are considered rigid and nonconducting and the proper physical boundary conditions in this case are

$$\tilde{v} = 0, \qquad \frac{\partial \tilde{p}}{\partial y} = 0, \qquad \frac{\partial \tilde{\rho}}{\partial y} = 0 \tag{7}$$

The axial velocity $\tilde{u}$ cannot be given along a rigid boundary or the initial boundary-value problem will in general be overspecified. The second component of Eq. (3a) was integrated along the wall to obtain $\tilde{u}$ in the current work.

Numerical Solution Method

Having developed boundary conditions for the acoustic problem, attention is now focused on the numerical solution technique. One specific algorithm that has gained wide use and acceptance for solving time-dependent problems in fluid dynamics was developed by MacCormack.[13] This algorithm has been shown to give consistent reliable results when applied to the acoustic system[7] and was used here to integrate both the differential equations and boundary conditions. During the boundary condition integration, all normal gradients were approximated using one-sided differences before performing the integration. Details of the application of MacCormack's algorithm are given in Ref. 13 and are not discussed here.

Transient Filter

Spectral analysis is widely used to determine the frequency content in measured acoustic time histories.[14] It will demonstrated in the following that it can also provide an invaluable tool for eliminating or filtering transient waves captured by numerical schemes. The process of obtaining a series of discrete numerical values for an acoustic field at a point in space is known as sampling. Here MacCormack's method will be thought of as a means for sampling the acoustic field; it reproduces the total field, both transient and steady state, at discrete instants of time. Because the frequency content of the steady-state field is known, it is advantageous to use spectral analysis and a filter to eliminate the components of the field at frequencies which do not contribute to the steady state. Such a process is referred to here as a transient filter.

The specific procedure utilized in this work is a straightforward application of standard Fast Fourier Transform (FFT) techniques. Starting from a particular initial state, the total acoustic field $\{\tilde{\Phi}(x,y,t)\}$ is sampled at 512 or 2^9 instants of time utilizing MacCormack's method. The time series is then transformed into discrete frequency components by use of the FFT calculation. This yields the complex frequency components $\{\hat{\Phi}(x,y,\omega)\}$ of the total field. An example of the results of this transform will be given later in terms of acoustic pressure by displaying the square of the magnitude of its frequency components on a decibel scale:

$$dB = 20 \log_{10} [\hat{p}(x,y,\omega)\hat{p}^*(x,y,\omega)] \tag{8}$$

Here the asterisk denotes the complex conjugate and dB is the acoustic pressure spectrum in decibels. An approximation to the periodic steady-state field is then obtained by filtering the components at the transient frequencies from $\{\hat{\Phi}\}$ and inverting the transform; i.e., by performing the inverse FFT on the spectrum including only the components at the harmonics of the fundamental driving source frequency. Specific details on the implementation of the FFT are available in Ref. 14.

Two-Step Iterative Method

The solution obtained by MacCormack's method can be combined with the transient filter to obtain the steady state to a specified accuracy. The following procedure, referred to here as a "two-step iterative method" is proposed:

Step 1

1a) Starting with an arbitrary initial condition, integrate the acoustic system coupled with the boundary conditions forward in time for a specified number of timesteps.

1b) Save the acoustic field at each timestep.

Step 2

2a) Perform an FFT on the data obtained from step 1b and filter by removing contributions at all frequencies except the fundamental source frequency and its harmonics.

2b) Perform the inverse FFT on results of step 2a to obtain an approximation to the steady state.

2c) Evaluate the approximation steady-state solution obtained from step 2b at the largest time in the interval and use this result as the new initial condition for step 1a.

This two-step iterative method is performed until the steady state is obtained to a desired accuracy, i.e., until the results of step 2c at iteration N differ by less than a prescribed small amount from those at iteration N-1.

Results and Discussion

The integrity of the two-step method is tested by attempting to reproduce the exact periodic steady-state solution for a benchmark linear problem. The steady-state solution for a nonlinear acoustic problem is also obtained with the method. Results are computed with 51 evenly spaced points in both the x and y direction in a square channel in which $L = H = \pi c_0/1320$. The benchmark problem chosen was that of a low-amplitude monochromatic acoustic source radiating into a quiescent no-flow channel with a pressure release (i.e., $z = 0$) inflow boundary condition. This problem has the advantage that an exact solution can be written which includes both the transient and the steady-state waves, and it is a case in which significant reflections exist in the solution domain.

The exact solution for the disturbance pressure of the benchmark problem will be a solution to the linearized wave equation and can be expressed in the form

$$\tilde{p}(x,y,t) = P^s(x,y,t) + P^t(x,y,t) \tag{9a}$$

$$P^t = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{H} \sin \gamma_{mn} t \tag{9b}$$

$$P^s = \sum_{m=0}^{\infty} C_m \frac{\sin K_m x}{\sin K_m L} \cos \frac{m\pi y}{H} \sin \tilde{\omega} t \tag{9c}$$

$$\gamma_{mn} = c_0 \sqrt{\left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \tag{9d}$$

$$K_m = \sqrt{k^2 - \left(\frac{m\pi}{H}\right)^2} \tag{9e}$$

$$A_{mn} = \frac{-2(-1)^n (kL)(n\pi) C_m}{[(K_m L)^2 - (n\pi)^2] \sqrt{(n\pi)^2 + \left(\frac{m\pi L}{H}\right)^2}} \tag{9f}$$

In Eqs. (9), $P^s(x,y,t)$ and $P^t(x,y,t)$ correspond to the steady-state and transient wave fields, where $\tilde{\omega}$ is the source circular frequency $2\pi/T$. The C_m are the known modal amplitudes of the source pressure, and $k = \tilde{\omega}/c$.

To simulate the pressure release inflow boundary condition numerically, the following values of the three critical inflow parameters were used:

$$b(y) = 0, \qquad \zeta = 0, \qquad z = 0$$

The solution was advanced forward 512 timesteps at each spatial location before applying the transient filter. During the transient filter, the frequency spectrum was divided into 512 evenly spaced points each $\tilde{\omega}/5$ increments apart. Henceforth an iteration shall mean an advancement of the solution 512 timesteps using MacCormack's method followed by the transient filter. The ambient state is considered to be uniform with $p_0 = 101{,}325$ N/m^2, $c_0 = 343$ m/s, and $\rho_0 = 1.2$ kg/m^3. The driving source frequency is taken to be $1100/\pi$ Hz (i.e., $\tilde{\omega} = 2\pi f = 2200$), and the solution is calculated for two cases, $m = 0$ and $m = 1$, one mode at a time.

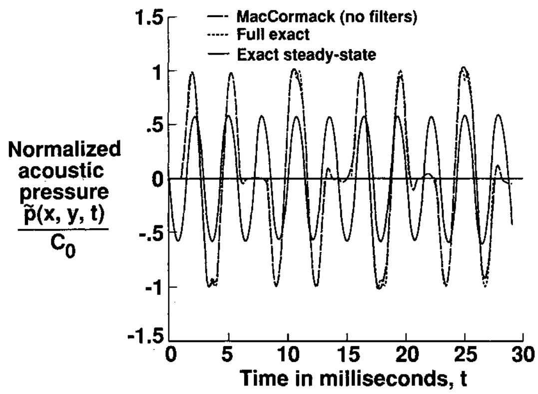

Fig. 2 Plane wave acoustic pressure ($x = L/2$, $y = H/4$).

Normalized acoustic pressure-time histories for a plane wave source are compared to the exact solution given by Eq. (9a) in Fig. 2. The source was considered to be a plane wave of the form

$$p_s(y,t) = C_0 \sin \tilde{\omega} t, \qquad C_0 = 10^{-3} p_0$$

Time histories were normalized with the source pressure amplitude C_0. The curve corresponding to the MacCormack results was computed without applying the transient filter. Although the acoustic pressure is compared only at the spatial location $x = L/2$ and $y = H/4$ in the figure, these results were typical of those at 40 other spatial locations examined at random in the channel. The solid line in the figure is the exact steady-state solution P^s given by Eq. (9), and the full exact solution depicted in the figure was computed with only the first 20 transient modes (i.e., $n = 1, 2, 3, \ldots, 20$) in the series. It is apparent that the MacCormack solution is in good agreement with the full exact solution. There is, however, a small discrepancy between the exact and MacCormack solutions near the peak amplitudes. Experimentation with the number of transient modes in the exact series expansion indicates that this difference is eliminated by increasing the number of transient modes included in the exact series. The most important point to be inferred from this curve is that the full solution given by the MacCormack method is significantly different from that of the exact steady state. Furthermore, no matter how far in time the acoustic system is integrated, no steady state is obtained. Such a situation exists because the transient acoustic wave is always present in the solution, and no numerical technique can, in the presence of nondecaying reflections, reproduce the steady-state solution unless the initial conditions can be chosen to coincide exactly with it.

The acoustic pressure spectrum expressed in decibels (i.e., dB) provides insight into how acoustic energy is distributed throughout the frequency spectrum and was computed here prior to applying the transient filter. Figure 3 depicts dB at two representative locations, ($3L/4$, $H/4$) and ($L/2$, $H/4$) after 512 timesteps. The component of the energy at frequency $\tilde{\omega}$ represents the contribution from the steady state whereas the contributions at other frequencies are due to the transient wave field. The conclusion that can be drawn from this plot is that acoustic energy is spread over the entire frequency spectrum. It is clear from this figure that at some locations the transient carries a significant portion of the total acoustic energy.

The ability of the two-step method to converge to a steady state for plane acoustic waves was tested by performing the second step of the two-step method (i.e., the transient filter). To establish the credibility of the method, an exhaustive study was performed using up to ten iterations on the two-step method. Approximately 40 spatial locations within the domain were investigated at random for convergence to a steady state. Results in Fig. 4 for which $x = L/4$ and $y = H/4$ are

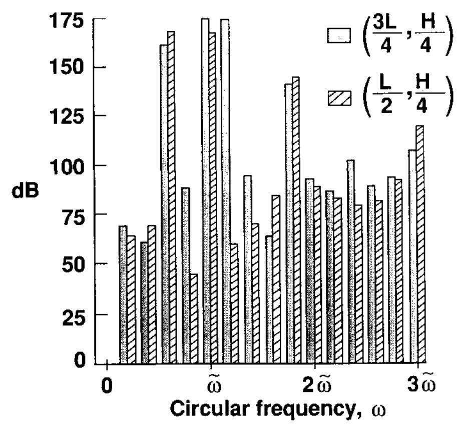

Fig. 3 Pressure spectrum in decibels for a plane wave.

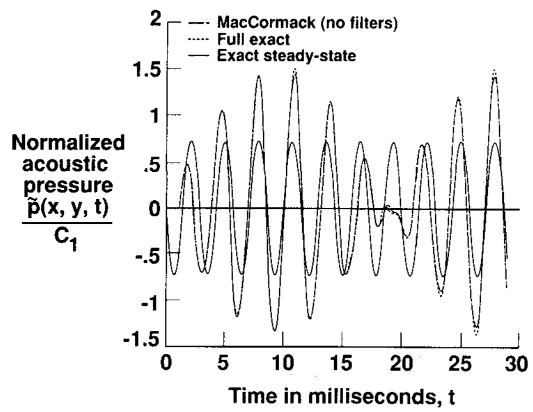

Fig. 6 Acoustic pressure ($m = 1$, $x = L/2$, $y = H/4$).

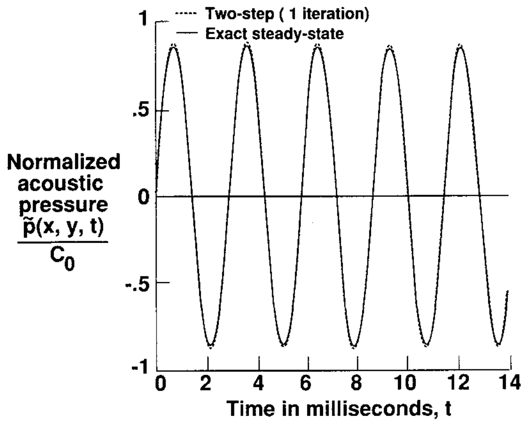

Fig. 4 Steady-state acoustic pressure ($m = 0$).

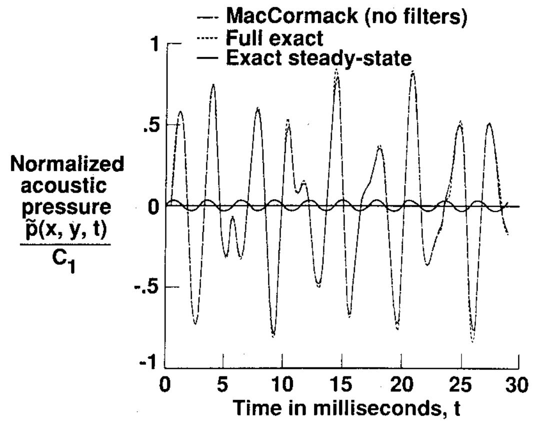

Fig. 7 Acoustic pressure ($m = 1$, $x = 3L/4$, $y = H/4$).

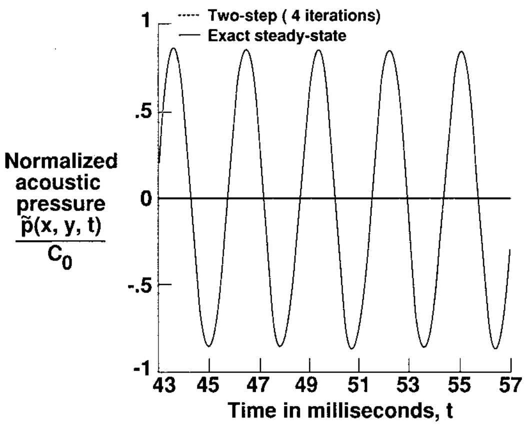

Fig. 5 Steady-state acoustic pressure ($m = 0$).

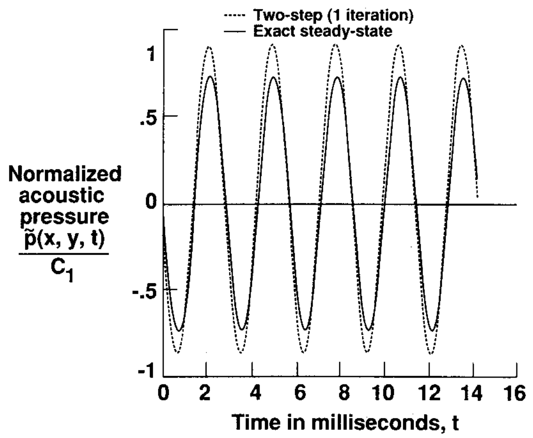

Fig. 8 Acoustic pressure ($m = 1$, $x = L/2$, $y = H/4$).

representative of results obtained after a single iteration on the two-step method. The exact steady-state solution shown in the figure was taken from Eq. (9c) and all acoustic pressure fluctuations were normalized with C_0. As shown in the figure, after a single iteration of the two-step method results are nearly undistinguishable from the exact steady state. Results after four iterations are shown in Fig. 5 and similar results to those of Fig. 5 were obtained after ten iterations. It is clear from these calculations that the two-step method is remarkably stable for planar acoustic sources.

Results were also computed for a nonplanar (i.e., $m = 1$) acoustic source. The source pressure was chosen as

$$p_s(y,t) = C_1 \cos(\pi y/H) \sin \tilde{\omega} t, \qquad C_1 = 10^{-3} p_0$$

Again, approximately 40 spatial locations within the domain were investigated at random so that any pitfalls in the current method could be highlighted and all time histories were normalized with the source pressure amplitude C_1. At all spatial points within the domain, the method yielded the exact steady state in either two or four iterations. Figure 6 shows results obtained at the spatial location $(L/2,\ H/4)$ before applying the transient filter. Results in this figure are representative of those obtained for the category of spatial points requiring only two iterations on the two-step method before evolving to a steady state. Points in this category are characterized by the fact that the exact solution and the steady-state solution are of the same order of magnitude. MacCormack's method and the exact solution are in good agreement and the small discrepancies at the peak amplitudes are a result of taking only 20 terms (i.e., $n = 20$) in Eq. (9b). Figure 7 shows results obtained at the spatial location ($3L/4$, $H/4$) before applying the transient filter. Results in this figure are representative of the category of spatial points requiring four iterations on the two-step method before converging to a steady state. Here the steady-state solution is two to three orders of magnitude smaller than the full solution obtained before applying the transient filter. Note also that the MacCormack time histories shown in Figs. 6 and 7 are significantly different than the exact steady-state solutions in the figures. The transient filter will be required to obtain the correct steady state.

The two-step method was applied to the time histories computed by MacCormack's method for the nonplanar source. Results after one and two iterations on the time histories of Fig. 6 are shown in Figs. 8 and 9, respectively. As shown in Fig. 8, after a single iteration of the two-step method only the peak amplitudes have not converged to the exact steady state. After two iterations (Fig. 9) results cannot be distinguished from the exact steady state. The two-step method has also been applied to the MacCormack time histories computed in Fig. 7. Approximate steady-state solutions obtained after one and four iterations are plotted in Figs. 10 and 11, respectively. After one iteration (Fig. 10), the numerical solution shows poor agreement with the exact steady-state solution. Although the results are not shown, after two iterations the agreement with the exact steady state was considerably improved. Finally, after the fourth iteration (Fig. 11), results cannot be distinguished from the exact steady state. Results in Figs. 10 and 11 indicate that even when the steady-state wave is several orders of magnitude smaller than the transient wave, a large number of iterations on the two-step method is not required to obtain a steady state.

Finally, a nonlinear steady-state solution will be obtained. For simplicity, nonlinear results here are restricted to a plane wave source of the form

$$p_s(y,t) = C_0 \sin \tilde{\omega} t$$

To simulate nonlinear acoustic propagation the source amplitude C_0 was chosen as $0.2p_0$ and $0.3p_0$. Calculations were performed for up to 40 iterations on the two-step method. Results within the channel were observed to be independent of the transverse coordinate y and all pressure-time histories were normalized with the source amplitude C_0. Figure 12 shows results after two iterations on the two-step method. Time histories in the figures represent the centerline (i.e., $x = L/2$, $y = H/2$) values and are representative of several other locations investigated at random within the channel. The solid curve in each figure depicts the exact solution from linear theory and is included as a reference against which to compare the two nonlinear cases.

Both the shape and peak amplitude of the time history for the lower amplitude case ($C_0 = 0.2p_0$) did not change beyond two iterations, evidence that a steady state has been achieved for this case. However, the peak amplitude of the time history for the higher-amplitude case ($C_0 = 0.2p_0$) was observed to change from -0.8 after two iterations to a value of -1.3 after the sixth iteration, evidence that a steady state was not reached after two iterations. Figure 13 shows the computed time histories for the higher-amplitude source at 40 iterations. Both the peak amplitude and shape of the time history did not change from those at the sixth iteration. Thus, even the higher-amplitude case has reached a steady state after six

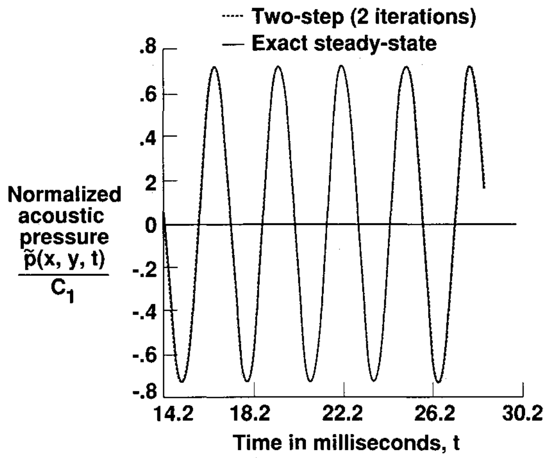

Fig. 9 Acoustic pressure ($m = 1$, $x = L/2$, $y = H/2$).

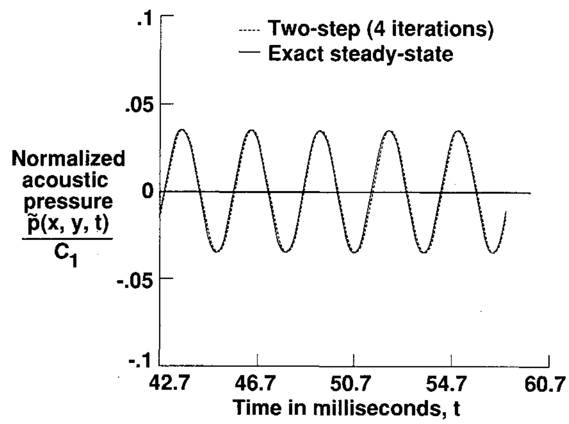

Fig. 11 Acoustic pressure ($m = 1$, $x = 3L/4$, $y = H/4$).

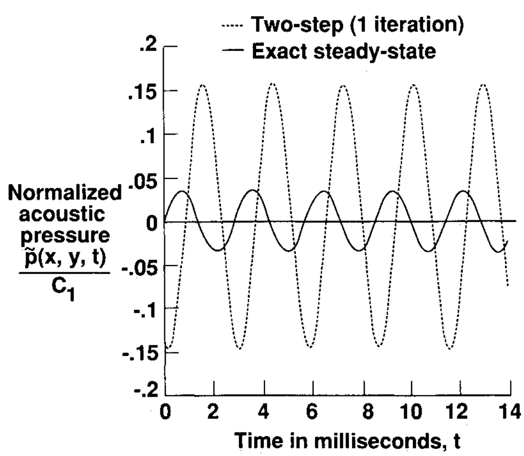

Fig. 10 Acoustic pressure ($m = 1$, $x = 3L/4$, $y = H/4$).

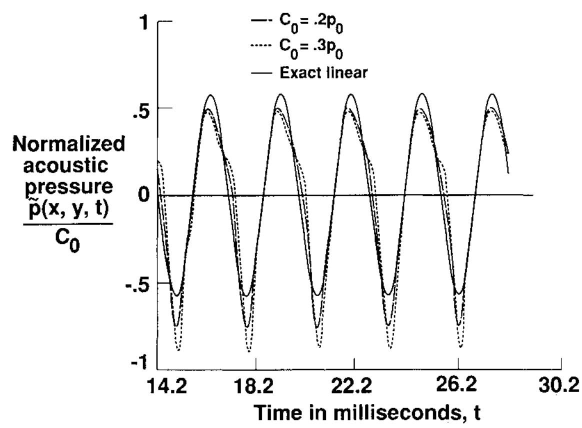

Fig. 12 Steady-state solution (two iterations, $m = 0$).

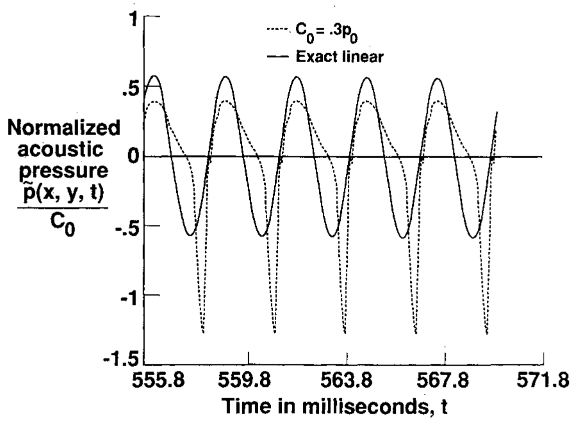

Fig. 13 Steady-state solution (40 iterations, $m = 0$).

iterations. It is emphasized that these nonlinear results cannot be obtained analytically and, so far as the authors are aware, they are new.

Conclusions

The analysis presented in this paper represents a new approach for obtaining numerical solutions to periodic steady-state acoustic problems. The method, referred to as a two-step method, has been tested on a benchmark acoustic problem for which an exact analytical solution is available. Results were also calculated for a nonlinear acoustic problem for which an exact analytical solution is not known. Based upon the results of this study the following conclusions are drawn:

1) The two-step method represents a powerful, efficient, and stable method for evolving two-dimensional acoustic systems to a periodic steady state.

2) The method is applicable to any number of spatial dimensions and to other hyperbolic systems.

3) For the benchmark problem only a single iteration on the method is required when the transient and steady-state field are of the same order of magnitude. However, four iterations are required when the steady-state field is several orders of magnitude smaller than the transient field.

4) The method requires six iterations before achieving a steady state for the nonlinear test problem.

5) The method eliminates the necessity to develop nonreflecting boundary conditions to obtain steady-state solutions for higher-dimensional acoustic problems and may provide the only alternative for obtaining steady-state solutions to problems for which nonreflecting boundary conditions are physically incorrect.

It should be noted that the sample solutions presented in this paper are of limited scope, and further calculations need to be made to evaluate the two-step method with flow and in the presence of more complicated geometries and boundary conditions.

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